ON SOME WEIGHTED OSTEOWSKI INEQUALITEIS

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ABSTRACT The aim of this note is to establish new weighted Ostrowski like inequalities.

Key Words Ostrowski inequality, Č EBYŠEV functional, Gruss inequality

INTRODUCTION

In 1938, A. M. Ostrowski $[3-4]$ proved the following classical inequality:

Theorem 1 Let $f:[a,b]\to \mathbb{R}$ be continuos on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b)\to \mathbb{R}$ is bounded on (a,b) i.e., $|f'(x)|\le M<\infty$. Then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)M, \tag{1.1}$$

for all $x\in[a,b]$, where M is constant.

For two absolutely continuous functions $f,g:[a,b]\to \mathbb{R}$, consider the functional

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \tag{1}$$

provided, the involved integrals exist.

In 1882, P. L. Č ebyšev $[1]$ proved that, if $f',g'\in L_\infty[a,b]$, then

$$T(f,g)\le\frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty. \tag{2}$$

In 1934, G. Grüss $[2]$ showed that

$$T(f,g)\le\frac{1}{4}(M-m)(N-n), \tag{3}$$

provided m, M, n and N are real numbers satisfying the condition

$$-\infty<m\le f(x)\le M<\infty,$$
$$-\infty<n\le g(x)\le N<\infty,$$

for all $x\in[a,b]$.

In the last few years, the study of integral inequalities has been disscused by many mathematicians and a number of research papers have published which deal with various generalizations and extensions, see $[5-9]$ and references given therein. Inspired and motivated by the above said facts going on related to inequalities (1), (3) and (4), we establish here new weighted Ostrowski type inequalities for the product of two continuous functions whose first derivatives are in $\mathbf{L}_\infty(a,b)$.

MAIN RESULTS

Let the weight $w:[a,b]\to 0,\infty)$, be non-negative, integrable and

$$\int_a^b w(t)dt<\infty.$$

The domain of w may be finite or infinite. We denote the zero moment as

$$m(a,b)=\int_a^b w(t)dt.$$

Our main results are as follows.

Theorem 2 Let $f,g:\ \to \mathbb{R}$ be continuous functions on $[a,b]$ and differentiable on (a,b), whose derivatives $f',g':\to \mathbb{R}$ are bounded on (a,b), i.e., $\|f'\|_\infty=\sup_{x\in(a,b)}|f'(x)|<\infty$, $\|g'\|_\infty=\sup_{x\in(a,b)}|g'(x)|<\infty$. Then

$$|f(x)g(x)-\frac{1}{2m(a,b)}(g(x)\int_a^b w(y)f(y)dy+f(x)\int_a^b w(y)g(y)dy)|$$
$$\le\frac{1}{2m(a,b)}(|g(x)|\|f'\|_\infty+|f(x)|\|g'\|_\infty)\int_a^b w(y)|x-y|dy$$

for all $x\in a,b]$.

Proof Following the approach of [6], for any $x,y\in a,b]$ we have the following identities:

$$f(x)-f(y)=\int_y^x f'(t)dt, \tag{4}$$

$$g(x)-g(y)=\int_y^x g'(t)dt. \qquad (5)$$

Multiplying both sides of (5) and (6) by $g(x)$ and $f(x)$ respectively and adding we get

$$2f(x)g(x)-(f(y)g(x)+f(x)g(y))=g(x)\int_y^x f'(t)dt+f(x)\int_y^x g'(t)dt. \qquad (6)$$

Now multiplying both sides of (7) with $w(y)$ and integrating both sides with respect to y over $[a,b]$, then rewriting we have

$$f(x)g(x)-\frac{1}{2m(a,b)}\left(g(x)\int_a^b w(y)f(y)dy+f(x)\int_a^b w(y)g(y)dy\right)$$

$$=\frac{1}{2m(a,b)}\int_a^b w(y)\left(g(x)\int_y^x f'(t)dt+f(x)\int_y^x g'(t)dt\right)dy,$$

implies

$$\left|f(x)g(x)-\frac{1}{2m(a,b)}\left(g(x)\int_a^b w(y)f(y)dy+f(x)\int_a^b w(y)g(y)dy\right)\right|$$

$$=\left|\frac{1}{2m(a,b)}\int_a^b w(y)\left(g(x)\int_y^x f'(t)dt+f(x)\int_y^x g'(t)dt\right)dy\right|$$

$$\le\frac{1}{2m(a,b)}\int_a^b w(y)\left(|g(x)|\left|\int_y^x |f'(t)|dt\right|+|f(x)|\left|\int_y^x |g'(t)|dt\right|\right)dy$$

$$\le\frac{1}{2m(a,b)}\int_a^b w(y)\left(|g(x)|\|f'\|_\infty|x-y|+|f(x)|\|g'\|_\infty|x-y|\right)dy$$

$$=\frac{1}{2m(a,b)}\left(|g(x)|\|f'\|_\infty+|f(x)|\|g'\|_\infty\right)\int_a^b w(y)|x-y|dy.$$

Hence proved.

Remark 1 Multiplying both side of (8) by $\frac{w(x)}{m(a,b)}$ and integrating with respect to x over $[a,b]$, rewriting the identity and using the properties of modulus, we obtain the following inequality:

$$|T(w,f,g)|\le\frac{1}{2m^2(a,b)}\int_a^b w(x)\left(\int_a^b w(y)\left(|g(x)|\|f'\|_\infty+|f(x)|\|g'\|_\infty\right)|x-y|dy\right)dx,$$

where

$$T(w,f,g)=\frac{1}{m(a,b)}\int_a^b w(t)f(t)g(t)dt-\left(\frac{1}{m(a,b)}\int_a^b w(t)f(t)dt\right)\left(\frac{1}{m(a,b)}\int_a^b w(t)g(t)dt\right). \qquad (7)$$

Theorem 3 Let f,g,f',g' be as in Theorem 1, then

$$\left|f(x)g(x)-\frac{1}{m(a,b)}\left(g(x)\int_a^b w(y)f(y)dy+f(x)\int_a^b w(y)g(y)dy\right)+\frac{1}{m(a,b)}\int_a^b w(y)f(y)g(y)dy\right|$$

$$\le\frac{1}{m(a,b)}\|f'\|_\infty.\|g'\|_\infty\int_a^b w(y)|x-y|^2\,dy.$$

Proof From the hypothesis, the identities (5) and (6) hold. Multiplying the left and right sides of (5) and (6) we get

$$f(x)g(x)-\left(f(y)g(x)+f(x)g(y)\right)+f(y)g(y)=\left(\int_y^x f'(t)dt\right)\left(\int_y^x g'(t)dt\right).$$

Now multiplying both sides of (10) with $w(y)$, integrating both sides with respect to y over $[a,b]$ and rewriting we have

$$f(x)g(x)-\frac{1}{m(a,b)}\left(g(x)\int_a^b w(y)f(y)dy+f(x)\int_a^b w(y)g(y)dy\right)$$

$$+\frac{1}{m(a,b)}\int_a^b w(y)f(y)g(y)dy$$

$$= \frac{1}{m(a,b)} \int_a^b w(y) \left(\int_y^x f'(t)dt \right) \left(\int_y^x g'(t)dt \right) dy, \qquad (10)$$

implies

$$| f(x)g(x) - \frac{1}{m(a,b)} \begin{pmatrix} g(x)\int_a^b w(y) f(y)dy + f(x) \\ \int_a^b w(y)g(y)dy \end{pmatrix} +$$

$$\frac{1}{m(a,b)} \int_a^b w(y) f(y)g(y)dy \,|$$

$$= \frac{1}{m(a,b)} \left| \int_a^b w(y) \left(\int_y^x f'(t)dt \right) \left(\int_y^x g'(t)dt \right) dy \right|$$

$$\le \frac{1}{m(a,b)} \int_a^b w(y) \left(\int_y^x |f'(t)|dt \right) \left(\int_y^x |g'(t)|dt \right) dy$$

$$\le \frac{1}{m(a,b)} \int_a^b w(y) \left(\| f' \|_\infty |x-y| \right) \left(\| g' \|_\infty |x-y| \right) dy$$

$$= \frac{1}{m(a,b)} \| f' \|_\infty . \| g' \|_\infty \int_a^b w(y) |x-y|^2 \, dy,$$

and the proof is completed.

Remark 2 Multiplying both side of (11) by $\frac{w(x)}{m(a,b)}$ and integrating with respect to x over $[a,b]$, rewriting the identity and using the properties of modulus, we obtain the following inequality:

$$|T(w,f,g)| \le \frac{1}{2m^2(a,b)} \| f' \|_\infty \| g' \|_\infty$$

$$\int_a^b w(x) \left(\int_a^b w(y) |x-y|^2 \, dy \right) dx.$$

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